The Fokker-Planck Equation and the First Exit Time Problem. A Fractional Second Order Approximation

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1 The Stochastic Model

Following an approach in stochastic analysis we assume that the state $S = S(t)$ of an individual part of a stochastic system at time t follows a stochastic process of the form:

$$dS(t) = h(t)dt + \sigma(t)dW(t), \tag{1}$$

where $h(t)$ is the drift coefficient or the infinitesimal mean and $\sigma(t)$ the variance parameter or the infinitesimal variance or the diffusion coefficient and $W(t)$ the standard Wiener process. The latter is a good alternative to reproduce a stochastic process of Brownian motion type that is a random process to account for the random changes of our system. Accordingly an equation of the last type can model the time course of a complex system as are several complicated machines or automata. The last equation is immediately integrable provided we have selected appropriate initial conditions as $S(t = 0) = S(0)$.

$$S(t) = S(0) + \int_0^t h(s)ds + \int_0^s \sigma(s)dW(s), \tag{2}$$

This equation form gives a large number of stochastic paths for the health state $S(t)$ of the individual parts of the system. However, it should be noted that these paths following a random process with drift are artificial realizations that can not calculated in the real life for a specific part of the system else we have a perfect

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inspection system estimating the state of the particles in real time; that is impossible so far. However, we can find methods to estimate special characteristics of this system of stochastic paths provided by the last equation as is the summation of infinitesimal mean that is the mean value $H(t)$ of the systems state over time given by

$$H(t) = S(0) + \int_0^t h(s)ds, \tag{3}$$

Even so it is not feasible to calculate $H(t)$ immediately. Fortunately there are several theoretical approaches to find the time development of $H(t)$ from the advances in physics, mathematics and applied mathematics, and probability and statistics. The first approach is by observing that as $H(t)$ expresses the State of a large ensemble of the system's elements, it should be a declining function of time or better age.

2 General Solution

To find the appropriate form of $H(t)$ a series of delicate mathematical calculations are needed. We preferred estimates leading to closed form solutions thus providing important and easy applied tools for scientists from several fields and those non familiar with stochastic theory methodology and practice. The important steps leading to the final forms for estimating $H(t)$ are given in the following.

The first step is the formulation of the transition probability density function $p(S, t)$, that is a function expressing the probability for the health state S of an individual to move from one point at time t to the next. This is achieved by calculating a Chapman-Kolmogorov equation for the discrete case. However, we use here the continuous alternative of this equation that is the following Fokker-Planck equation:

$$\frac{\partial p(S,t)}{\partial t} = -h(t)\frac{\partial [p(S,t)]}{\partial S} + \frac{1}{2}[\sigma(t)]^2 \frac{\partial^2 [p(S,t)]}{\partial S^2}, \tag{4}$$

The related information can be found in Adriaan Fokker doctoral dissertation in Leiden University in 1913 [1] and in his paper 1914 [2]. Fokker in his disertation presents the related equation in connection with Max Planck theory in a specific chapter. Latter on Max Plack presented the Fokker-Planck equation in his paper in 1917 [4].

2.1 *Fractional Forms of the Fokker-Planck Equation*

The main fractional forms of the Fokker-Planck equation are the following

$$\frac{\partial p(S,t)}{\partial t} = -h(t)\frac{\partial [p(S,t)]}{\partial S} + \frac{1}{2}[\sigma(t)]^2 \frac{\partial^\alpha [p(S,t)]}{\partial S^\alpha}, \tag{5}$$

where α is the fractional parameter

In several problems in physics and engineering it is preferred to set $D = \frac{1}{2}\sigma^2$ to obtain the form

$$\frac{\partial p(S,t)}{\partial t} = -h(t)\frac{\partial [p(S,t)]}{\partial S} + D\frac{\partial^a [p(S,t)]}{\partial S^a}, \tag{6}$$

The selection of the diffusion parameter D became famous from the Einstein paper in a case of diffusion problem with zero drift with a non-fractional (classical) equation of the form

$$\frac{\partial p(S,t)}{\partial t} = D\frac{\partial^2 [p(S,t)]}{\partial S^2}, \tag{7}$$

The other fractional form of the Fokker-Planck equation refers to a fractional derivative with respect to time

$$\frac{\partial^\gamma p(S,t)}{\partial^\gamma t} = -h(t)\frac{\partial [p(S,t)]}{\partial S} + \frac{1}{2}[\sigma(t)]^2\frac{\partial [p(S,t)]}{\partial S}, \tag{8}$$

where γ is the fractional parameter.

The literature for the solution of these fractional Fokker-Planck equations is already quite large (see Sun et al. [12]). A significant part of this literature is included in the chapters of this book along with important applications.

2.2 Solution of the Fokker-Planck Equation

The solution of the classical Fokker-Planck partial differential equation is highly influenced by the boundary conditions selected. Here this partial differential equation for S and t is solved for the following appropriate boundary conditions (see Janssen and Skiadas [3])

$$p(S(t), t_0; S_0, t_0) = \delta(S(t) - S_0), \tag{9}$$

$$\frac{\partial p[S(t), t_0; S_0, t]}{\partial S(t)} \to 0 \quad \text{as} \quad S(t) \to \pm\infty \tag{10}$$

For the solution we use the method of characteristic functions. The characteristic function $\phi(S,t)$ is introduced by the following equation

$$\phi(S,t) = \int_{-\infty}^{+\infty} p(S,t;S_0,t_0)\exp(isS)ds, \tag{11}$$

Integrating by parts and using the Fokker-Plank equation we arrive at

$$\frac{\partial \phi}{\partial t} = ish(t)\phi - \frac{1}{2}[\sigma(t)]^2 s^2 \phi, \tag{12}$$

which with the initial conditions proposed

$$\phi(s, t_0) = \exp(isS_0), \tag{13}$$

is solved providing the following expression for ϕ

$$\phi(s, t_0) = \exp\left[is\left[S_0 + \int_{t_0}^{t} h(t')dt'\right] - \frac{1}{2}s^2 \int_{t_0}^{t} \left[\sigma(t')\right]^2 dt'\right], \tag{14}$$

This is the characteristic function of a Gaussian with mean

$$\left[S_0 + \int_{t_0}^{t} h(t')dt'\right], \tag{15}$$

and variance

$$\left[\sigma(t')\right]^2 dt', \tag{16}$$

Considering Eq. 3 and $t_0 = 0$ the solution is

$$p(t) = \frac{1}{[2\pi \int_0^t [\sigma(s)]^2 ds]^{1/2}} \exp\left[-\frac{[H(t)]^2}{2\int_0^t [\sigma(s)]^2 ds}\right], \tag{17}$$

3 Specific Solution

When $\sigma(t) = \sigma$ a simple presentation of the transition probability density function during time is given by:

$$p(t) = \frac{1}{\sigma\sqrt{2\pi t}} \exp\left[-\frac{[H(t)]^2}{2\sigma^2 t}\right], \tag{18}$$

Having estimated the transition probability density function for the continuous process we can find the first exit time probability density function for the process reaching a barrier.

4 A First Approximation Form

Just after the introduction of the Fokker-Planck equation in 1913 it was possible to estimate the first exit or hitting time of a stochastic process from a barrier. Schrödinger [5] and Smoluchowsky [11] solved the problem publishing two independent papers in the same journal issue in 1915. The drift should be linear of the form

$$H(t) = l - bt, \tag{19}$$

where l and b are parameters. Then the resulting distribution function $g(t)$ is known as the Inverse Gaussian of the form

$$g(t) = \frac{l}{t} p(t) = \frac{l}{t} \frac{1}{\sigma\sqrt{2\pi t}} \exp\left[-\frac{[l - bt)]^2}{2\sigma^2 t}\right], \tag{20}$$

Figure 1 (left) illustrates the Inverse Gaussian model. The linear drift is presented by a red heavy line whereas the stochastic paths appear as light lines. The confidence intervals are also presented by dashed lines. The inverse Gaussian Distribution is illustrated in Fig. 1 (right) as well as a fit curve to data (Data from the Carey Medflies).

The Inverse Gaussian as expressed in the last equation is a convenient form to find a generalization for a smooth nonlinear drift $H(t)$. From Fig. 2 we consider a tangent approximation in the point $M(H, t)$ of the drift curve $H(t)$.

$$l(t) = H(t) - tH'(t), \tag{21}$$

Note that the minus sign $(-)$ accounts for the negative slope of the derivative $H'(t)$. Now the only needed is to replace l by $l(t)$ and $l - bt$ by $H(t)$ in the formula of the Inverse Gaussian to obtain to tangent or first approximation for the first exit time density of a general smooth drift.

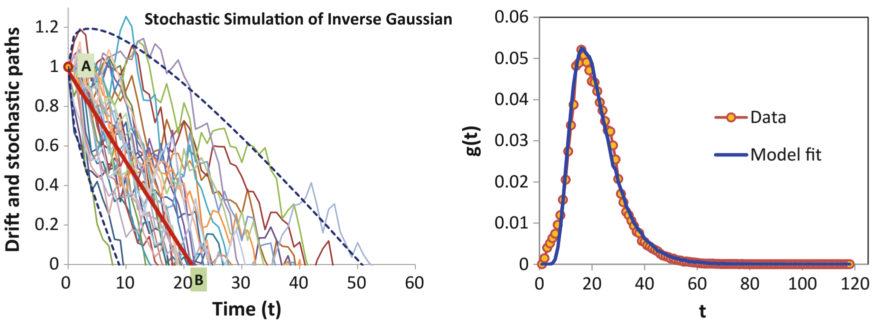

Fig. 1 (Left) Presentation of the inverse Gaussian with stochastic paths and (right) the inverse Gaussian distribution. Data from Carey Medflies

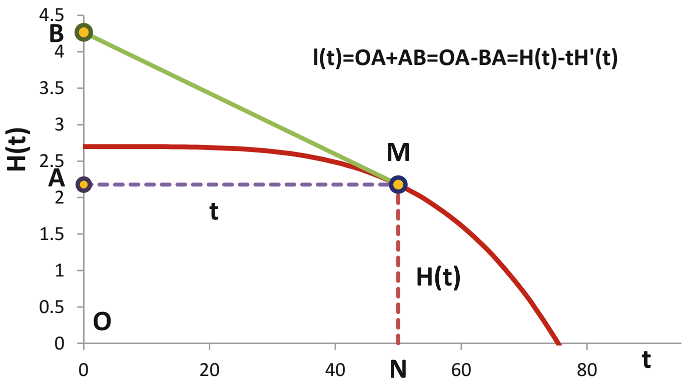

Fig. 2 Generalization of the inverse Gaussian

$$g(t) = \frac{|H - tH'|}{t} p(t), \tag{22}$$

By using the estimated $p(t)$ we arrive at the following form for the first exit time probability density function

$$g(t) = \frac{|H - tH'|}{\sigma\sqrt{2\pi t^3}} \exp\left[-\frac{[H(t)]^2}{2\sigma^2 t}\right], \tag{23}$$

The last formula is coming from a first approximation of the first exit time densities with good results in relatively simpler cases (see Skiadas and Skiadas [6–8]).

5 A Second Order Fractional Correction

Clearly the first order approximation error is smaller as the the drift $H(t)$ approaches linearity. For the other cases a second order approximation is needed by means of taking into account the second order derivative or even higher order derivatives. However, the second order derivative approach could be a good approximation provided that a fraction of this derivative is selected to account of the smaller or larger curvature of the drift $H(t)$. The resulting formula is the following

$$g(t) = \frac{1}{\sigma\sqrt{2\pi t}} \left[\frac{|H - tH'|}{t} + k\frac{t^2}{2}\frac{H''}{|H - tH'|}\right] \exp\left[-\frac{[H(t)]^2}{2\sigma^2 t}\right], \tag{24}$$

where the parameter k expresses the fraction of the second derivative needed. We take the quadratic term of a Taylor series expansion for $H(t)$ that is

$$H(t) = H(0) + tH' + \frac{t^2}{2}H'' + \cdots , \tag{25}$$

Evenmore the first order approximation $|H - tH'|$ is used as a normalising factor for the quadratic term.

5.1 An Interesting Application

We can arrive in a very interesting formula by selecting the following form for $H(t)$:

$$H(t) = l - (bt)^c, \tag{26}$$

where l, b, c are parameters

In our applications $H(t)$ expresses the health state of a population and thus the first exit time distribution $g(t)$ refers to the age of the population by means of $0 \leq t$. Accordingly the $g(t)$ is a half distribution and the resulting first and second order approximations are:

$$g(t) = \frac{2|l + (c-1)(bt)^c|}{\sigma\sqrt{2\pi t^3}} \exp\left[-\frac{[l-(bt)^c]^2}{2\sigma^2 t}\right], \tag{27}$$

$$g(t) = \frac{2}{\sigma\sqrt{2\pi t}}\left[\frac{|l + (c-1)(bt)^c|}{t} + \frac{kt^2 c(c-1)b^c t^{(c-2)}}{2|l + (c-1)(bt)^c|}\right] \exp\left[-\frac{[l-(bt)^c]^2}{2\sigma^2 t}\right], \tag{28}$$

As for fitting this formula and the previous simpler forms to data sets it is not possible to estimate the parameters of the model along with σ, two options are selected; that is to set $\sigma = 1$ and estimate l, b, c or to set $l = 1$ and estimate b, c, σ. The latter is very important when stochastic simulations are needed. It is also useful for applications on health state estimates. It was selected from Weitz and Fraser [13] for the application in Medflies and from Skiadas and Skiadas [6–10] for many applications.

The applications are done in actual data (mortality) sets or on a logarithmic transformation of the data providing better information for the first period of the human lifespan. The first order approximation (see Fig. 3) fail to cover the first period of the lifespan where strongly nonlinear forms appear. Instead, the fractional approximations with the second derivative (see Fig. 4) provide very good fitting to the data sets.

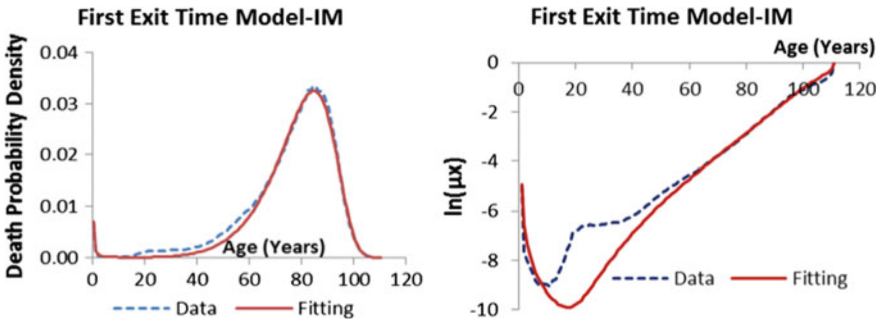

Fig. 3 (Left) First exit time distribution (data and fit curves for USA mortality data, male 2010) and (right) respective application with a logarithmic form of the data. First order approximation

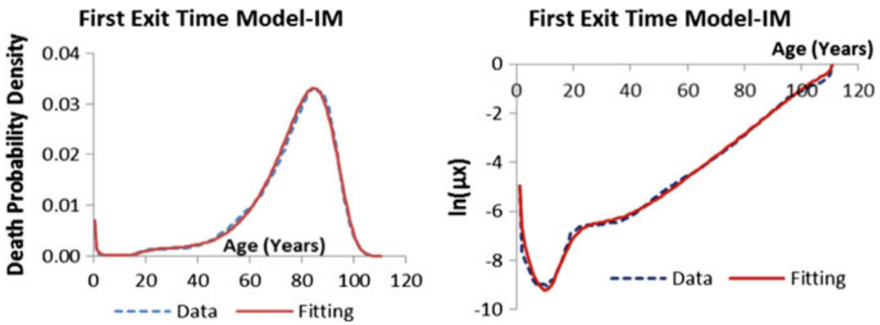

Fig. 4 (Left) First exit time distribution (data and fit curves for USA mortality data, male 2010) and (right) respective application with a logarithmic form of the data. Second order approximation

6 Summary and Conclusions

We have present a first exit time theory of a stochastic process. The general model is analytically derived according to the first exit time or hitting time theory for a stochastic process crossing a barrier. The derivation lines follow the transition probability densities from the Fokker-Planck equation. Then we find the probability density form and the first and second approximation of the first exit time densities. For the first approximation we obtain a generalization of the Inverse Gaussian whereas for the second approximation we apply a fractional approach to the second derivative by introducing a parameter k. We thus introduce another approach to apply a fractional theory. Instead to apply the fractional derivative theory to the Fokker-Plank equation we have solved a classical Fokker-Plank equation and then we have selected a fractional approach when introducing the second order derivative.

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